

Heat transfer in a complex medium

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Abstract

The heat equation is considered in the complex medium consisting of many small bodies (particles) embedded in a given material. On the surfaces of the small bodies an impedance boundary condition is imposed. An equation for the limiting field is derived when the characteristic size a of the small bodies tends to zero, their total number $\mathcal{N}(a)$ tends to infinity at a suitable rate, and the distance $d = d(a)$ between neighboring small bodies tends to zero: $a \ll d$, $\lim_{a \rightarrow 0} \frac{a}{d(a)} = 0$. No periodicity is assumed about the distribution of the small bodies. These results are basic for a method of creating a medium in which heat signals are transmitted along a given line. The technical part for this method is based on an inverse problem of finding potential with prescribed eigenvalues.

Keywords:

heat transfer; many-body problem; transmission of heat signals; inverse problems; materials science.

MSC 80M40; 80A20, 35B99; 35K20; 35Q41; 35R30; 74A30; 74G75

PACS 65.80.-g

1 Introduction and results

In this paper the problem of heat transfer in a complex medium consisting of many small impedance particles of an arbitrary shape is solved. Equation for the effective limiting temperature is derived when the characteristic size a of the particles tends to zero while their number tends to infinity at a suitable rate while the distance d between closest neighboring particles is much larger than a , $d \gg a$.

These results are used for developing a method for creating materials in which heat is transmitted along a line. Thus, the information can be transmitted by a heat signals.

The contents of this paper is based on the earlier papers of the author cited in the bibliography, especially [6], [18] and [19].

Let many small bodies (particles) D_m , $1 \leq m \leq M$, of an arbitrary shape be distributed in a bounded domain $D \subset \mathbb{R}^3$, $\text{diam} D_m = 2a$, and the boundary

of D_m is denoted by \mathcal{S}_m and is assumed twice continuously differentiable. The small bodies are distributed according to the law

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_{\Delta} N(x) dx [1 + o(1)], \quad a \rightarrow 0. \quad (1)$$

Here $\Delta \subset D$ is an arbitrary open subdomain of D , $\kappa \in [0, 1)$ is a constant, $N(x) \geq 0$ is a continuous function, and $\mathcal{N}(\Delta)$ is the number of the small bodies D_m in Δ . The heat equation can be stated as follows:

$$u_t = \nabla^2 u + f(x) \text{ in } \mathbb{R}^3 \setminus \bigcup_{m=1}^M D_m, \quad u|_{t=0} = 0, \quad (2)$$

$$u_N = \zeta_m u \text{ on } \mathcal{S}_m, \quad 1 \leq m \leq M, \quad \operatorname{Re} \zeta_m \geq 0. \quad (3)$$

Here N is the outer unit normal to \mathcal{S} ,

$$\mathcal{S} := \bigcup_{m=1}^M \mathcal{S}_m, \quad \zeta_m = \frac{h(x_m)}{a^\kappa}, \quad x_m \in D_m, \quad 1 \leq m \leq M,$$

and $h(x)$ is a continuous function in D , $\operatorname{Re} h \geq 0$.

Denote

$$\mathcal{U} := \mathcal{U}(x, \lambda) = \int_0^\infty e^{-\lambda t} u(x, t) dt.$$

Then, taking the Laplace transform of equations (2) - (3) one gets:

$$-\nabla^2 \mathcal{U} + \lambda \mathcal{U} = \lambda^{-1} f(x) \text{ in } \Omega, \quad (4)$$

$$\mathcal{U}_N = \zeta_m \mathcal{U} \text{ on } \mathcal{S}_m, \quad 1 \leq m \leq M. \quad (5)$$

Let

$$g(x, y) := g(x, y, \lambda) := \frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi|x-y|}, \quad (6)$$

$$F(x, \lambda) := \frac{1}{\lambda} \int_{\mathbb{R}^3} g(x, y) f(y) dy. \quad (7)$$

Look for the solution to (4) - (5) of the form

$$\mathcal{U}(x, \lambda) = F(x, \lambda) + \sum_{m=1}^M \int_{\mathcal{S}_m} g(x, s) \sigma_m(s) ds, \quad (8)$$

where

$$\mathcal{U}(x, \lambda) := \mathcal{U}(x) := \mathcal{U}, \quad (9)$$

and $\mathcal{U}(x)$ depends on λ .

The functions σ_m are unknown and should be found from the boundary conditions (5). Equation (4) is satisfied by \mathcal{U} of the form (8) with arbitrary

continuous σ_m . To satisfy the boundary condition (5) one has to solve the following equation obtained from the boundary condition (5):

$$\frac{\partial \mathcal{U}_e(x)}{\partial N} + \frac{A_m \sigma_m - \sigma_m}{2} - \zeta_m \mathcal{U}_e - \zeta_m T_m \sigma_m = 0 \text{ on } \mathcal{S}_m, \quad 1 \leq m \leq M, \quad (10)$$

where the effective field $\mathcal{U}_e(x)$ is defined by the formula:

$$\mathcal{U}_e(x) := \mathcal{U}_{e,m}(x) := \mathcal{U}(x) - \int_{\mathcal{S}_m} g(x, s) \sigma_m(s) ds, \quad (11)$$

the operator T_m is defined by the formula:

$$T_m \sigma_m = \int_{\mathcal{S}_m} g(s, s') \sigma_m(s') ds', \quad (12)$$

and A_m is:

$$A_m \sigma_m = 2 \int_{\mathcal{S}_m} \frac{\partial g(s, s')}{\partial N_s} \sigma_m(s') ds'. \quad (13)$$

In deriving equation (10) we have used the known formula for the outer limiting value on \mathcal{S}_m of the normal derivative of a simple layer potential.

We now apply the ideas and methods for solving many-body scattering problems developed in [6] - [9].

Let us call $\mathcal{U}_{e,m}$ the effective (self-consistent) value of \mathcal{U} , acting on the m -th body. As $a \rightarrow 0$, the dependence on m disappears, since

$$\int_{\mathcal{S}_m} g(x, s) \sigma_m(s) ds \rightarrow 0 \text{ as } a \rightarrow 0.$$

One has

$$\mathcal{U}(x, \lambda) = F(x, \lambda) + \sum_{m=1}^M g(x, x_m) Q_m + \mathcal{J}_2, \quad x_m \in D_m, \quad (14)$$

where

$$Q_m := \int_{\mathcal{S}_m} \sigma_m(s) ds, \quad (15)$$

$$\mathcal{J}_2 := \sum_{m=1}^M \int_{\mathcal{S}_m} [g(x, s') - g(x, x_m)] \sigma_m(s') ds'.$$

Define

$$\mathcal{J}_1 := \sum_{m=1}^M g(x, x_m) Q_m. \quad (16)$$

We prove in Lemma 3, Section 4 (see also [6] and [18]) that

$$|\mathcal{J}_2| \ll |\mathcal{J}_1| \text{ as } a \rightarrow 0 \quad (17)$$

provided that

$$\lim_{a \rightarrow 0} \frac{a}{d(a)} = 0, \quad (18)$$

where $d(a) = d$ is the minimal distance between neighboring particles.

If (17) holds, then problem (4) - (5) is solved asymptotically by the formula

$$\mathcal{U}(x, \lambda) = F(x, \lambda) + \sum_{m=1}^M g(x, x_m) Q_m, \quad a \rightarrow 0, \quad (19)$$

provided that asymptotic formulas for Q_m , as $a \rightarrow 0$, are found.

To find formulas for Q_m , let us integrate (10) over \mathcal{S}_m , estimate the order of the terms in the resulting equation as $a \rightarrow 0$, and keep the main terms, that is, neglect the terms of higher order of smallness as $a \rightarrow 0$.

We get

$$\int_{\mathcal{S}_m} \frac{\partial \mathcal{U}_e}{\partial N} ds = \int_{D_m} \nabla^2 \mathcal{U}_e dx = O(a^3). \quad (20)$$

Here we assumed that $|\nabla^2 \mathcal{U}_e| = O(1)$, $a \rightarrow 0$. This assumption is valid since $\mathcal{U} = \lim_{a \rightarrow 0} \mathcal{U}_e$ is smooth as a solution to an elliptic equation. One has

$$\int_{\mathcal{S}_m} \frac{A_m \sigma_m - \sigma_m}{2} ds = -Q_m[1 + o(1)], \quad a \rightarrow 0. \quad (21)$$

This relation is proved in Lemma 2, Section 4, see also [6]. Furthermore,

$$-\zeta_m \int_{\mathcal{S}_m} \mathcal{U}_e ds = -\zeta_m |\mathcal{S}_m| \mathcal{U}_e(x_m) = O(a^{2-\kappa}), \quad a \rightarrow 0, \quad (22)$$

where $|\mathcal{S}_m| = O(a^2)$ is the surface area of \mathcal{S}_m . Finally,

$$\begin{aligned} -\zeta_m \int_{\mathcal{S}_m} ds \int_{\mathcal{S}_m} g(s, s') \sigma_m(s') ds' &= -\zeta_m \int_{\mathcal{S}_m} ds' \sigma_m(s') \int_{\mathcal{S}_m} ds g(s, s') \\ &= Q_m O(a^{1-\kappa}), \quad a \rightarrow 0. \end{aligned} \quad (23)$$

Thus, the main term of the asymptotic of Q_m , as $a \rightarrow 0$, is

$$Q_m = -\zeta_m |\mathcal{S}_m| \mathcal{U}_e(x_m). \quad (24)$$

Formulas (24) and (19) yield

$$\mathcal{U}(x, \lambda) = F(x, \lambda) - \sum_{m=1}^M g(x, x_m) \zeta_m |\mathcal{S}_m| \mathcal{U}_e(x_m, \lambda), \quad (25)$$

and

$$\mathcal{U}_e(x_m, \lambda) = F(x_m, \lambda) - \sum_{m' \neq m, m'=1}^M g(x_m, x_{m'}) \zeta_{m'} |\mathcal{S}_{m'}| \mathcal{U}_e(x_{m'}, \lambda). \quad (26)$$

Denote

$$\mathcal{U}_e(x_m, \lambda) := \mathcal{U}_m, \quad F(x_m, \lambda) := F_m, \quad g(x_m, x_{m'}) := g_{mm'},$$

and write (26) as a linear algebraic system for \mathcal{U}_m :

$$\mathcal{U}_m = F_m - a^{2-\kappa} \sum_{m' \neq m} g_{mm'} h_{m'} c_{m'} \mathcal{U}_{m'}, \quad 1 \leq m \leq M, \quad (27)$$

where $h_{m'} = h(x_{m'})$, $\zeta_{m'} = \frac{h_{m'}}{a^\kappa}$, $c_{m'} := |S_{m'}|a^{-2}$.

Consider a partition of the bounded domain D , in which the small bodies are distributed, into a union of $P \ll M$ small nonintersecting cubes Δ_p , $1 \leq p \leq P$, of side b ,

$$b \gg d, \quad b = b(a) \rightarrow 0 \quad \text{as } a \rightarrow 0 \quad \lim_{a \rightarrow 0} \frac{d(a)}{b(a)} = 0.$$

Let $x_p \in \Delta_p$, $|\Delta_p|$ = volume of Δ_p . One has

$$\begin{aligned} a^{2-\kappa} \sum_{m'=1, m' \neq m}^M g_{mm'} h_{m'} c_{m'} \mathcal{U}_{m'} &= a^{2-\kappa} \sum_{p'=1, p' \neq p}^P g_{pp'} h_{p'} c_{p'} \mathcal{U}_{p'} \sum_{x_{m'} \in \Delta_{p'}} 1 = \\ &= \sum_{p' \neq p} g_{pp'} h_{p'} c_{p'} \mathcal{U}_{p'} N(x_{p'}) |\Delta_{p'}| [1 + o(1)], \quad a \rightarrow 0. \end{aligned} \quad (28)$$

Thus, (27) yields a linear algebraic system (LAS) of order $P \ll M$ for the unknowns \mathcal{U}_p :

$$\mathcal{U}_p = F_p - \sum_{p' \neq p, p'=1}^P g_{pp'} h_{p'} c_{p'} N_{p'} \mathcal{U}_{p'} |\Delta_{p'}|, \quad 1 \leq p \leq P. \quad (29)$$

Since $P \ll M$, the order of the original LAS (27) is drastically reduced. This is crucial when the number of particles tends to infinity and their size a tends to zero. We have assumed that

$$h_{m'} = h_{p'} [1 + o(1)], \quad c_{m'} = c_{p'} [1 + o(1)], \quad \mathcal{U}_{m'} = \mathcal{U}_{p'} [1 + o(1)], \quad a \rightarrow 0, \quad (30)$$

for $x_{m'} \in \Delta_{p'}$. This assumption is justified, for example, if the functions $h(x)$, $\mathcal{U}(x, \lambda)$,

$$c(x) = \lim_{x_{m'} \in \Delta_x, a \rightarrow 0} \frac{|S_{m'}|}{a^2},$$

and $N(x)$ are continuous, but these assumptions can be relaxed.

The continuity of the $\mathcal{U}(x, \lambda)$ is a consequence of the fact that this function satisfies elliptic equation, and the continuity of $c(x)$ is assumed. If all the small bodies are identical, then $c(x) = c = \text{const}$, so in this case the function $c(x)$ is certainly continuous.

The sum in the right-hand side of (29) is the Riemannian sum for the integral

$$\lim_{a \rightarrow 0} \sum_{p'=1, p' \neq p}^P g_{pp'} h_{p'} c_{p'} N(x_{p'}) \mathcal{U}_{p'} |\Delta'_p| =$$

$$\int_D g(x, y) h(y) c(y) N(y) \mathcal{U}(y, \lambda) dy$$

Therefore, linear algebraic system (29) is a collocation method for solving integral equation

$$\mathcal{U}(x, \lambda) = F(x, \lambda) - \int_D g(x, y) c(y) h(y) N(y) \mathcal{U}(y, \lambda) dy. \quad (31)$$

Convergence of this method for solving equations with weakly singular kernels is proved in [10], see also [11] and [12].

Applying the operator $-\nabla^2 + \lambda$ to equation (31) one gets an elliptic differential equation:

$$(-\Delta + \lambda) \mathcal{U}(x, \lambda) = \frac{f(x)}{\lambda} - c(x) h(x) N(x) \mathcal{U}(x, \lambda). \quad (32)$$

Taking the inverse Laplace transform of this equation yields

$$u_t = \Delta u + f(x) - q(x)u, \quad q(x) := c(x) h(x) N(x). \quad (33)$$

Therefore, the limiting equation for the temperature contains the term $q(x)u$. Thus, the embedding of many small particles creates a distribution of source and sink terms in the medium, the distribution of which is described by the term $q(x)u$.

If one solves equation (31) for $\mathcal{U}(x, \lambda)$, or linear algebraic system (29) for $\mathcal{U}_p(\lambda)$, then one can Laplace-invert $\mathcal{U}(x, \lambda)$ for $\mathcal{U}(x, t)$. Numerical methods for Laplace inversion from the real axis are discussed in [13] - [14].

If one is interested only in the average temperature, one can use the relation

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T u(x, t) dt = \lim_{\lambda \rightarrow 0} \lambda \mathcal{U}(x, \lambda). \quad (34)$$

Relation (34) is proved in Lemma 1, Section 4. It holds if the limit on one of its sides exists. The limit on the right-hand side of (34) let us denote by $\psi(x)$. From equations (7) and (31) it follows that ψ satisfies the equation

$$\psi = \varphi - B\varphi,$$

where

$$\varphi := \int_{\Omega} g_0(x, y) f(y) dy,$$

$$g_0(x, y) := \frac{1}{4\pi|x - y|},$$

$$B\psi := \int_{\Omega} g_0(x, y)q(y)\psi(y)dy,$$

and

$$q(x) := c(x)h(x)N(x).$$

The function ψ can be calculated by the formula

$$\psi(x) = (I + B)^{-1}\varphi. \quad (35)$$

From the physical point of view the function $h(x)$ is non-negative because the flux $-\nabla u$ of the heat flow is proportional to the temperature u and is directed along the outer normal N : $-u_N = h_1 u$, where $h_1 = -h < 0$. Thus, $q \geq 0$.

It is proved in [15] - [16] that zero is not an eigenvalue of the operator $-\nabla^2 + q(x)$ provided that $q(x) \geq 0$ and

$$q = O\left(\frac{1}{|x|^{2+\epsilon}}\right), \quad |x| \rightarrow \infty,$$

and $\epsilon > 0$.

In our case, $q(x) = 0$ outside of the bounded region D , so the operator $(I + B)^{-1}$ exists and is bounded in $C(D)$.

Let us formulate our basic result.

Theorem 1. *Assume (1), (18), and $h \geq 0$. Then, there exists the limit $\mathcal{U}(x, \lambda)$ of $\mathcal{U}_\epsilon(x, \lambda)$ as $\epsilon \rightarrow 0$, $\mathcal{U}(x, \lambda)$ solves equation (31), and there exists the limit (34), where $\psi(x)$ is given by formula (35).*

Methods of our proof of Theorem 1 are quite different from the proof of homogenization theory results in [1] and [3].

The author's plenary talk at Chaos-2015 Conference was published in [20].

2 Creating materials which allows one to transmit heat signals along a line

In applications it is of interest to have materials in which heat propagates along a line and decays fast in all the directions orthogonal to this line.

In this Section a construction of such material is given. We follow [19] with some simplifications.

The idea is to create first the medium in which the heat transfer is governed by the equation

$$u_t = \Delta u - q(x)u \quad \text{in } D, \quad u|_S = 0, \quad u|_{t=0} = f(x), \quad (36)$$

where D is a bounded domain with a piece-wise smooth boundary S , $D = D_0 \times [0, L]$, $D_0 \subset \mathbb{R}^2$ is a smooth domain orthogonal to the axis x_1 , $x = (x_1, x_2, x_3)$, $x_2, x_3 \in D_0$, $0 \leq x_1 \leq L$.

Such a medium is created by embedding many small impedance particles into a given domain D filled with a homogeneous material. A detailed argument, given in Section 1 (see also [6] and [18]), yields the following result.

Assume that in every open subset Δ of D the number of small particles is defined by the formula:

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_{\Delta} N(x) dx [1 + o(1)], \quad a \rightarrow 0, \quad (37)$$

where $a > 0$ is the characteristic size of a small particle, $\kappa \in [0, 1)$ is a given number and $N(x) \geq 0$ is a continuous in D function.

Assume also that on the surface S_m of the m -th particle D_m the impedance boundary condition holds. Here

$$1 \leq m \leq M = \mathcal{N}(D) = O\left(\frac{1}{a^{2-\kappa}}\right), \quad a \rightarrow 0,$$

and the impedance boundary conditions are:

$$u_N = \zeta_m u \quad \text{on } S_m, \quad \operatorname{Re} \zeta_m \geq 0, \quad (38)$$

where

$$\zeta_m := \frac{h(x_m)}{a^\kappa}$$

is the boundary impedance, $x_m \in D_m$ is an arbitrary point (since D_m is small the position of x_m in D_m is not important), κ is the same parameter as in (37) and $h(x)$ is a continuous in D function, $\operatorname{Re} h \geq 0$, N is the unit normal to S_m pointing out of D_m . The functions $h(x)$, $N(x)$ and the number κ can be chosen as the experimenter wishes.

It is proved in Section 1 (see also [6], [18]) that, as $a \rightarrow 0$, the solution of the problem

$$u_t = \Delta u \quad \text{in } D \setminus \bigcup_{m=1}^M D_m, \quad u_N = \zeta_m u \quad \text{on } S_m, \quad 1 \leq m \leq M, \quad (39)$$

$$u|_S = 0, \quad (40)$$

and

$$u|_{t=0} = f(x), \quad (41)$$

has a limit $u(x, t)$. This limit solves problem (36) with

$$q(x) = c_S N(x) h(x), \quad (42)$$

where

$$c_S := \frac{|S_m|}{a^2} = \text{const}, \quad (43)$$

and $|S_m|$ is the surface area of S_m . By assuming that c_S is a constant, we assume, for simplicity only, that the small particles are identical in shape, see [6].

Since $N(x) \geq 0$ is an arbitrary continuous function and $h(x)$, $\text{Re} h \geq 0$, is an arbitrary continuous function, and both functions can be chosen by experimenter as he/she wishes, it is clear that an arbitrary real-valued potential q can be obtained by formula (42).

Suppose that

$$(-\Delta + q(x))\phi(x) = \lambda_n \phi_n, \quad \phi_n|_S = 0, \quad \|\phi_n\|_{L^2(D)} = \|\phi_n\| = 1, \quad (44)$$

where $\{\phi_n\}$ is an orthonormal basis of $L^2(D) := H$. Then the unique solution to (36) is

$$u(x, t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} (f, \phi_n) \phi_n(x). \quad (45)$$

If $q(x)$ is such that $\lambda_1 = 0$, $\lambda_2 \gg 1$, and $\lambda_2 \leq \lambda_3 \leq \dots$, then, as $t \rightarrow \infty$, the series (45) is well approximated by its first term

$$u(x, t) = (f, \phi_1) \phi_1 + O(e^{-10t}), \quad t \rightarrow \infty. \quad (46)$$

If $\lambda_1 > 0$ is very small, then the main term of the solution is

$$u(x, t) = (f, \phi_1) \phi_1 e^{-\lambda_1 t} + O(e^{-10t})$$

as $t \rightarrow \infty$. The term $e^{-\lambda_1 t} \sim 1$ if $t \ll \frac{1}{\lambda_1}$.

Thus, our problem is solved if $q(x)$ has the following property:

$$|\phi_1(x)| \text{ decays as } \rho \text{ grows,} \quad \rho = (x_2^2 + x_3^2)^{1/2}. \quad (47)$$

Since the eigenfunction is normalized, $\|\phi_1\| = 1$, this function will not tend to zero in a neighborhood of the line $\rho = 0$, so information can be transformed by the heat signals along the line $\rho = 0$, that is, along s -axis. Here we use the cylindrical coordinates:

$$x = (x_1, x_2, x_3) = (s, \rho, \theta), \quad s = x_1, \quad \rho = (x_2^2 + x_3^2)^{1/2}.$$

In Section 3 the domain D_0 is a disc and the potential $q(x)$ does not depend on θ .

The technical part of solving our problem consists of the construction of $q(x) = c_S N(x) h(x)$ such that

$$\lambda_1 = 0, \quad \lambda_2 \gg 1; \quad |\phi_1(x)| \text{ decays as } \rho \text{ grows.} \quad (48)$$

Since the function $N(x) \geq 0$ and $h(x)$, $\text{Re} h \geq 0$, are at our disposal, any desirable q , $\text{Re} q \geq 0$, can be obtained by embedding many small impedance particles in a given domain D . In Section 3, a potential q with the desired properties is constructed. This construction allows one to transform information along a straight line using heat signals.

3 Construction of $q(x)$

Let

$$q(x) = p(\rho) + Q(s),$$

where $s := x_1$, $\rho := (x_2^2 + x_3^2)^{1/2}$. Then the solution to problem (44) is $u = v(\rho)w(s)$, where

$$\begin{aligned} -v_m'' - \rho^{-1}v_m' + p(\rho)v_m &= \mu_m v_m, \quad 0 \leq \rho \leq R, \\ |v_m(0)| &< \infty, \quad v_m(R) = 0, \end{aligned} \quad (49)$$

and

$$\begin{aligned} -w_l'' + Q(s)w_l &= \nu_l w_l, \quad 0 \leq s \leq L, \\ w_l(0) &= 0, \quad w_l(L) = 0. \end{aligned} \quad (50)$$

One has

$$\lambda_n = \mu_m + \nu_l, \quad n = n(m, l). \quad (51)$$

Our task is to find a potential $Q(s)$ such that $\nu_1 = 0$, $\nu_2 \gg 1$ and a potential $p(\rho)$ such that $\mu_1 = 0$, $\mu_2 \gg 1$ and $|v_m(\rho)|$ decays as ρ grows.

It is known how to construct $q(s)$ with the desired properties: the Gel'fand-Levitan method allows one to do this, see [4]. Let us recall this construction. One has $\nu_{l0} = l^2$, where we set $L = \pi$ and denote by ν_{l0} the eigenvalues of the problem (50) with $Q(s) = 0$. Let the eigenvalues of the operator (50) with $Q \neq 0$ be $\nu_1 = 0$, $\nu_2 = 11$, $\nu_3 = 14$, $\nu_l = \nu_{l0}$ for $l \geq 4$.

The kernel $L(x, y)$ in the Gel'fand-Levitan theory is defined as follows:

$$L(x, y) = \int_{-\infty}^{\infty} \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} \frac{\sin(\sqrt{\lambda}y)}{\sqrt{\lambda}} d(\varrho(\lambda) - \varrho_0(\lambda)),$$

where $\varrho(\lambda)$ is the spectral function of the operator (50) with the potential $Q = Q(s)$, and $\varrho_0(\lambda)$ is the spectral function of the operator (50) with the potential $Q = 0$ and the same boundary conditions as for the operator with $Q \neq 0$.

Due to our choice of ν_l and the normalizing constants α_j , namely: $\alpha_j = \frac{\pi}{2}$ for $j \geq 2$ and $\alpha_1 = \frac{\pi^3}{3}$, the kernel $L(x, y)$ is given explicitly by the formula:

$$\begin{aligned} L(x, y) &= \frac{3xy}{\pi^3} + \frac{2}{\pi} \left(\frac{\sin(\sqrt{\nu_2}x)}{\sqrt{\nu_2}} \frac{\sin(\sqrt{\nu_2}y)}{\sqrt{\nu_2}} + \frac{\sin(\sqrt{\nu_3}x)}{\sqrt{\nu_3}} \frac{\sin(\sqrt{\nu_3}y)}{\sqrt{\nu_3}} \right) - \\ &\quad - \frac{2}{\pi} \left(\sin x \sin y + \sin(2x) \sin(2y) + \sin(3x) \sin(3y) \right), \end{aligned} \quad (52)$$

where $\nu_1 = 0$, $\nu_2 = 11$ and $\nu_3 = 14$. This is a finite rank kernel. The term xy is the value of the function $\frac{\sin \nu x}{\nu} \frac{\sin \nu y}{\nu}$ at $\nu = 0$, and the corresponding normalizing constant is $\frac{\pi^3}{3} = \|x\|^2 = \int_0^\pi x^2 dx$.

Solve the Gel'fand-Levitan equation:

$$K(s, \tau) + \int_0^s K(s, s')L(s', \tau)ds' = -L(s, \tau), \quad 0 \leq \tau \leq s, \quad (53)$$

which is uniquely solvable (see [4]). Since equation (53) has finite-rank kernel it can be solved analytically being equivalent to a linear algebraic system.

If the function $K(s, \tau)$ is found, then the potential $Q(s)$ is computed by the formula ([2], [4]):

$$Q(s) = 2 \frac{dK(s, s)}{ds}, \quad (54)$$

and this $Q(s)$ has the required properties: $\nu_1 = 0, \nu_2 \gg 1, \nu_l \leq \nu_{l+1}$.

Consider now the operator (49) for $v(\rho)$. Our problem is to calculate $p(\rho)$ which has the required properties:

$$\mu_1 = 0, \quad \mu_2 \gg 1, \quad \mu_m \leq \mu_{m+1},$$

and $|\phi_m(\rho)|$ decays as ρ grows.

We reduce this problem to the previous one that was solved above. To do this, set $v = \frac{\psi}{\sqrt{\rho}}$. Then equation

$$-v'' - \frac{1}{\rho}v' + p(\rho)v = \mu v,$$

is transformed to the equation

$$-\psi'' - \frac{1}{4\rho^2}\psi + p(\rho)\psi = \mu\psi. \quad (55)$$

Let

$$p(\rho) = \frac{1}{4\rho^2} + Q(\rho), \quad (56)$$

where $Q(\rho)$ is constructed above. Then equation (55) becomes

$$-\psi'' + Q(\rho)\psi = \mu\psi, \quad (57)$$

and the boundary conditions are:

$$\psi(R) = 0, \quad \psi(0) = 0. \quad (58)$$

The problem (57)- (58) has the desired eigenvalues $\mu_1 = 0, \mu_2 \gg 1, \mu_m \leq \mu_{m+1}$.

The eigenfunction

$$\phi_1(x) = v_1(\rho)w_1(s),$$

where $v_1(\rho) = \frac{\psi_1(\rho)}{\sqrt{\rho}}$, decays as ρ grows, and the eigenvalues λ_n can be calculated by the formula:

$$\lambda_n = \mu_m + \nu_l, \quad m, l \geq 1, \quad n = n(m, l).$$

Since $\mu_1 = \nu_1 = 0$ one has $\lambda_1 = 0$. Since $\nu_2 = 11$ and $\mu_2 = 11$, one has $\lambda_2 = 11 \gg 1$.

Thus, the desired potential is constructed:

$$q(x) = Q(s) + \left(\frac{1}{4\rho^2} + Q(\rho)\right),$$

where $Q(s)$ is given by formula (54).

This concludes the description of our procedure for the construction of q .

Remark 1. It is known (see, for example, [2]) that the normalizing constants

$$\alpha_j := \int_0^\pi \varphi_j^2(s) ds$$

and the eigenvalues λ_j , defined by the differential equation

$$-\frac{d^2\varphi_j}{ds^2} + Q(s)\varphi_j = \lambda_j\varphi_j,$$

the boundary conditions

$$\varphi_j'(0) = 0, \quad \varphi_j'(\pi) = 0,$$

and the normalizing condition $\varphi_j(0) = 1$, have the following asymptotic:

$$\alpha_j = \frac{\pi}{2} + O\left(\frac{1}{j^2}\right) \quad \text{as } j \rightarrow \infty,$$

and

$$\sqrt{\lambda_j} = j + O\left(\frac{1}{j}\right) \quad \text{as } j \rightarrow \infty.$$

The differential equation

$$-\Psi_j'' + Q(s)\Psi_j = \nu_j\Psi_j,$$

the boundary condition

$$\Psi_j(0) = 0, \quad \Psi_j(\pi) = 0,$$

and the normalizing condition $\Psi_j'(0) = 1$ imply

$$\sqrt{\lambda_j} = j + O\left(\frac{1}{j}\right) \quad \text{as } j \rightarrow \infty,$$

$$\Psi_j(s) \sim \frac{\sin(js)}{j} \quad \text{as } j \rightarrow \infty.$$

The main term of the normalized eigenfunction is:

$$\frac{\Psi_j}{\|\Psi_j\|} \sim \sqrt{2/\pi} \sin(js) \quad \text{as } j \rightarrow \infty,$$

and the main term of the normalizing constant is:

$$\alpha_j \sim \frac{\pi}{2j^2} \quad \text{as } j \rightarrow \infty.$$

4 Auxiliary results

Lemma 1 *If one of the limits $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t u(s) ds$ or $\lim_{\lambda \rightarrow 0} \lambda \mathcal{U}(\lambda)$ exists, then the other also exists and they are equal to each other:*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t u(s) ds = \lim_{\lambda \rightarrow 0} \lambda \mathcal{U}(\lambda),$$

where

$$\mathcal{U}(\lambda) := \int_0^\infty e^{-\lambda t} u(t) dt := \bar{u}(\lambda).$$

Proof. Denote

$$\frac{1}{t} \int_0^t u(t) dt := v(t), \quad \bar{u}(\sigma) := \int_0^\infty e^{-\sigma t} u(t) dt.$$

Then

$$\bar{v}(\lambda) = \int_\lambda^\infty \frac{\bar{u}(\sigma)}{\sigma} d\sigma$$

by the properties of the Laplace transform.

Assume that the limit $v(\infty) := v_\infty$ exists:

$$\lim_{t \rightarrow \infty} v(t) = v_\infty. \quad (59)$$

Then,

$$v_\infty = \lim_{\lambda \rightarrow 0} \lambda \int_0^\infty e^{-\lambda t} v(t) dt = \lim_{\lambda \rightarrow 0} \lambda \bar{v}(\lambda).$$

Indeed $\lambda \int_0^\infty e^{-\lambda t} dt = 1$, so

$$\lim_{\lambda \rightarrow 0} \lambda \int_0^\infty e^{-\lambda t} (v(t) - v_\infty) dt = 0,$$

and (59) is verified.

One has

$$\lim_{\lambda \rightarrow 0} \lambda \bar{v}(\lambda) = \lim_{\lambda \rightarrow 0} \int_\lambda^\infty \frac{\lambda}{\sigma} \bar{u}(\sigma) d\sigma = \lim_{\lambda \rightarrow 0} \lambda \bar{u}(\lambda), \quad (60)$$

as follows from a simple calculation:

$$\lim_{\lambda \rightarrow 0} \int_\lambda^\infty \frac{\lambda}{\sigma} \bar{u}(\sigma) d\sigma = \lim_{\lambda \rightarrow 0} \int_\lambda^\infty \frac{\lambda}{\sigma^2} \sigma \bar{u}(\sigma) d\sigma = \lim_{\sigma \rightarrow 0} \sigma \bar{u}(\sigma), \quad (61)$$

where we have used the relation $\int_\lambda^\infty \frac{\lambda}{\sigma^2} d\sigma = 1$.

Alternatively, let $\sigma^{-1} = \gamma$. Then,

$$\int_\lambda^\infty \frac{\lambda}{\sigma^2} \sigma \bar{u}(\sigma) d\sigma = \frac{1}{1/\lambda} \int_0^{1/\lambda} \frac{1}{\gamma} \bar{u}\left(\frac{1}{\gamma}\right) d\gamma = \frac{1}{\omega} \int_0^\omega \frac{1}{\gamma} \bar{u}\left(\frac{1}{\gamma}\right) d\gamma. \quad (62)$$

If $\lambda \rightarrow 0$, then $\omega = \lambda^{-1} \rightarrow \infty$, and if

$$\psi := \gamma^{-1} \bar{u}(\gamma^{-1}),$$

then

$$\lim_{\omega \rightarrow \infty} \frac{1}{\omega} \int_0^\omega \psi d\gamma = \psi(\infty) = \lim_{\gamma \rightarrow 0} \gamma^{-1} \bar{u}(\gamma^{-1}) = \lim_{\sigma \rightarrow 0} \sigma \bar{u}(\sigma). \quad (63)$$

Lemma 1 is proved. \square

Lemma 2 *Equation (21) holds.*

Proof. As $a \rightarrow 0$, one has

$$\frac{\partial}{\partial N_s} \frac{e^{-\sqrt{\lambda}|s-s'|}}{4\pi|s-s'|} = \frac{\partial}{\partial N_s} \frac{1}{4\pi|s-s'|} + \frac{\partial}{\partial N_s} \frac{e^{-\sqrt{\lambda}|s-s'|} - 1}{4\pi|s-s'|}. \quad (64)$$

It is known (see [5]) that

$$\int_{\mathcal{S}_m} ds \int_{\mathcal{S}_m} \frac{\partial}{\partial N_s} \frac{1}{4\pi|s-s'|} \sigma_m(s') ds' = -\frac{1}{2} \int_{\mathcal{S}_m} \sigma_m(s') ds' = -\frac{1}{2} Q_m. \quad (65)$$

On the other hand, as $a \rightarrow 0$, one has

$$\left| \int_{\mathcal{S}_m} ds \int_{\mathcal{S}_m} \frac{e^{-\sqrt{\lambda}|s-s'|} - 1}{4\pi|s-s'|} \sigma_m(s') ds' \right| \leq |Q_m| \int_{\mathcal{S}_m} ds \frac{1 - e^{-\sqrt{\lambda}|s-s'|}}{4\pi|s-s'|} = o(Q_m). \quad (66)$$

The relations (65) and (66) justify (21).

Lemma 2 is proved. \square

Lemma 3 *If assumption (18) holds, then inequality (17) holds.*

Proof. One has

$$\mathcal{J}_{1,m} := |g(x, x_m)Q| = \frac{|Q_m| e^{-\sqrt{\lambda}|x-x_m|}}{4\pi|x-x_m|}, \quad (67)$$

and

$$\mathcal{J}_{2,m} \leq \frac{e^{-\sqrt{\lambda}|x-x_m|}}{4\pi|x-x_m|} \max \left(\sqrt{\lambda}a, \frac{a}{|x-x_m|} \right) \int_{\mathcal{S}_m} |\sigma_m(s')| ds' \quad (68)$$

where $|x-x_m| \geq d$, and $d > 0$ is the smallest distance between two neighboring particles. One may consider only those values of λ for which $\lambda^{1/4}a < \frac{a}{d}$, because for the large values of λ , such that $\lambda^{1/4} \geq \frac{1}{d}$ the value of $e^{-\sqrt{\lambda}|x-x_m|}$ is negligibly small. The average temperature depends on the behavior of \mathcal{U} for small λ , see Lemma 1.

One has $|Q_m| = \int_{\mathcal{S}_m} |\sigma_m(s')| ds' > 0$ because σ_m keeps sign on \mathcal{S}_m , as follows from equation (24) as $a \rightarrow 0$.

It follows from (67) - (68) that

$$\left| \frac{\mathcal{J}_{2,m}}{\mathcal{J}_{1,m}} \right| \leq O\left(\left| \frac{a}{x - x_m} \right| \right) \leq O\left(\frac{a}{d} \right) \ll 1. \quad (69)$$

From (69) by the arguments similar to the given in [17] one obtains (17).

Lemma 3 is proved. \square

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